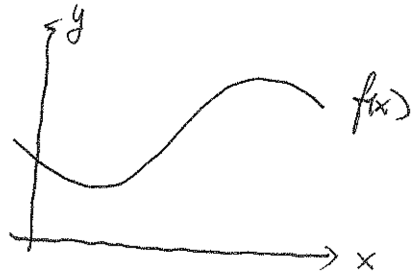
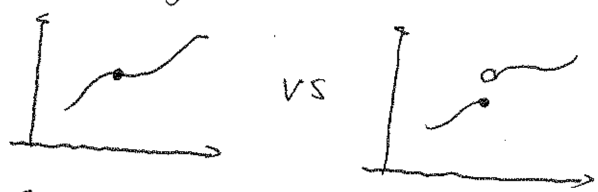
# Math 132. Overview.

- Function: $y = f(x)$
  - $y$ ← output
  - $x$ ← input.



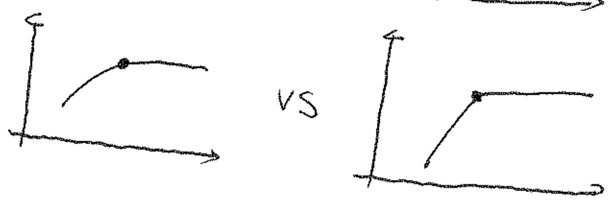

- Calculus
  - Differentiation: local behavior/properties
    subtraction/dividing
  - Integration: global behavior/properties
    addition / sum / union.

- Locally, we care about:
  - rate of change, slope of secant/tangent line
  - whether the curve is "smooth"




  etc.

- Globally, we are interested in:




  Summation/Area/Distance etc

- **Limit**: A concept to describe an "endless" approximation process.

  e.g. How to describe "as large (small) as possible"

  10, 100, 1000, 10000, .... or 0.1, 0.01, 0.001, ....

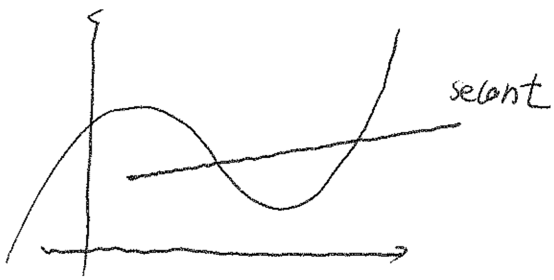
# §1.4 Tangent and Velocity

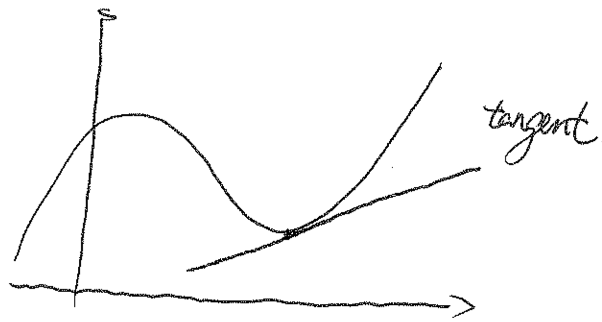
Key points: ① Secant / Tangent line (of a given curve) and their slopes.

② Average rate of change / Average velocity.

- Secant line: intersects the curve of $y=f(x)$ MORE THAN ONCE.

Tangent line: intersects the curve of $y=f(x)$ ONLY ONCE.

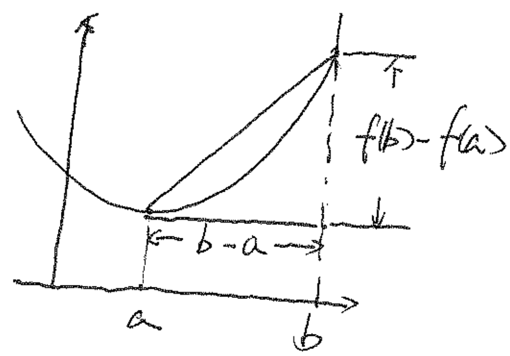



- Average rate of change of the function $y=f(x)$ over $x\in[a,b]$.

(*) $A.R.o.C := \dfrac{f(b)-f(a)}{b-a}$

= slope of the secant line passing through $(a, f(a))$ and $(b, f(b))$.




eg.1. Compute the average rate of change of $y=1+\sin x$ over $[0,\frac{\pi}{6}]$

Remark: Apply definition (*) to $a=0$, $b=\frac{\pi}{6}$ and $f(x)=1+\sin x$.

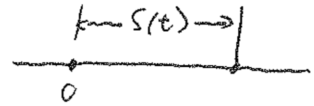
$$A.R.o.C. = \frac{(1+\sin\frac{\pi}{6})-(1+\sin 0)}{\frac{\pi}{6}-0}$$

$$= \frac{\sin\frac{\pi}{6}-\sin 0}{\frac{\pi}{6}}$$

Hint: $\sin\frac{\pi}{6}=\frac{1}{2}$, $\sin 0=0$

$$\boxed{= \frac{\frac{1}{2}}{\frac{\pi}{6}} = \frac{3}{\pi}}$$

- Consider a moving particle with displacement $s(t)$.




(*) Average velocity (over time interval $[t_1, t_2]$)

$$:= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

eg2 (F16). A particle moves according to the law of motion $s = t^3 - 6t^2 + 5t$, $t > 0$.
Find the average velocity over the interval $[0, 2]$

Solution: $s(0) = 0 - 0 + 0 = 0$, $s(2) = 2^3 - 6\cdot 2^2 + 5\cdot 2 = -6$

$$V_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{-6 - 0}{2 - 0} = -3$$

- eg3. Find the slope of the secant line of the function
$f(x) = x^2 + x$ on the interval $[1, 2]$

Remark: It is equivalent to ask for the ARoC of $f(x)$ over $[1, 2]$.

Solution: $\text{slope} = \frac{f(2) - f(1)}{2 - 1} = \frac{(2^2 + 2) - (1^2 + 1)}{2 - 1} = \frac{6 - 2}{1} = \boxed{4}$

- We are interested in when the secant line approaches the tangent line.

Roughly speaking, the slope of the tangent line can be approximated by the the slope of the secant line AS THE LENGTH OF THE INTERVAL APPROACHES ZERO.

e.g. 4. Find the slope of the secant line of $y = x^2 + x$ over $[1, 1+h]$ ($h$ is some small number)
And estimate the slope of the tangent line at $x = 1$.

Solution: slope of the secant line $= \frac{[(1+h)^2 + (1+h)] - [1^2 + 1]}{(1+h) - 1} = \frac{1 + 2h + h^2 + 1 + h - 2}{h}$

$$= \frac{h^2 + 3h}{h} = h + 3$$

As $h$ approaches 0,
the slope approaches $\boxed{3.}$ (slope of the tangent line)

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Remark for Webwork: Last part of ※3, ※4, ※5.

Take the average of the two slopes of the secant lines.

## §1.5 The limit of a function.

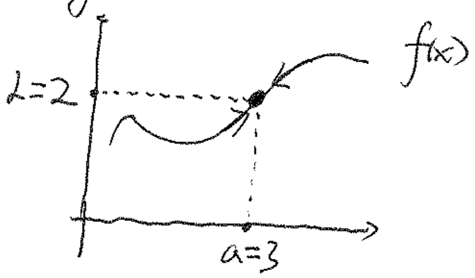
Key points: ① Intuitive idea of LIMIT from the graph of a function

② One-sided (left, right) / Two sided limits.

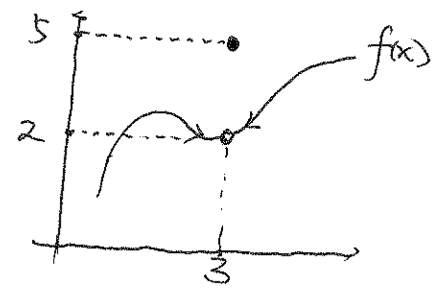
③ Infinite limits and VERTICAL ASYMPTOTES

- "Definition" of limit: If $f(x)$ approaches $L$ as $x$ approaches $a$, then we say $f(x)$ HAS LIMIT $L$ AT $x=a$. And write $\boxed{\lim\limits_{x\to a} f(x) = L}$
- Graphically, it means the following picture:

eg. 1.



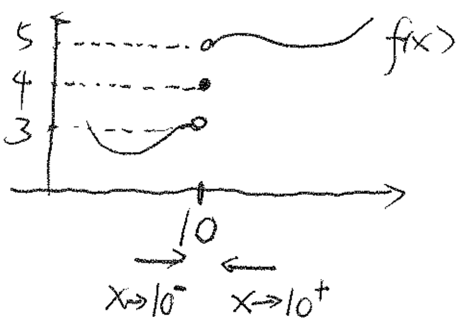

$$\lim_{x\to 3} f(x) = 2$$
($a$) ($L$)




- One-sided limit: If $f(x)$ approaches $L$ as $x$ approaches $a$ FROM THE LEFT (right) then $f(x)$ has LEFT (right) limit $L$ at $x=a$. We write

Left limit: $\lim\limits_{\boxed{x\to a^-}} f(x) = L$ ; Right limit $\lim\limits_{\boxed{x\to a^+}} f(x) = L$.

eg.2.




$\lim\limits_{x\to 10^+} f(x) = 5$, $\lim\limits_{x\to 10^-} f(x) = 3$

$f(10) = 4$.

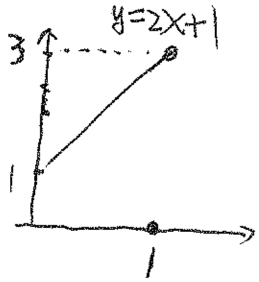
Remark: The limit of $f(x)$ at $a$ HAS NOTHING TO DO with the value of $f(x)$ at $a$, $f(a)$.

Remark: $\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = L$.

If $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, we say $\lim\limits_{x\to a} f(x)$ DOES NOT EXIST (DNE).

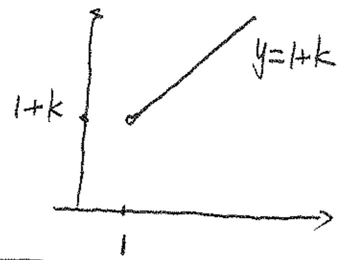
- e.g. 3. Let $f(x) = \begin{cases} 2x+1 & x<1 \\ 0 & x=1 \\ x+k & x>1 \end{cases}$ for some constant $k$ to be determined.

Find $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$. For what value of $k$ will the left and right limits of $f(x)$ at $x=1$ equal?




solution: $\lim\limits_{x\to 1^-} f(x) = 2\cdot 1 + 1 = 3$ since

$\lim\limits_{x\to 1^+} f(x) = 1+k$




$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x)$ if

$3 = 1+k$, i.e., $\boxed{k=2}$.

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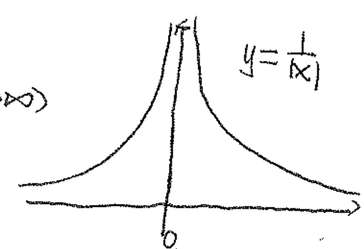
- Infinite limit: $\pm\infty$

Consider the function: $f(x) = \frac{1}{|x|}$. Recall the domain is $(-\infty, 0), (0, +\infty)$

We see $f(x)$ GETS ARBITRAILY LARGE as $x$ approaches 0.

In this case, we write: $\boxed{\lim\limits_{x\to 0} f(x) = \infty}$




vertical asymptote $x=0$

In general, $\lim\limits_{x\to a} f(x) = \infty$ indicates $f(x)$ can be made arbitrarily large (positive) as $x$ approaches $a$.

$\lim\limits_{x\to a} f(x) = -\infty$ indicates $f(x)$ can be made arbitrarily large negative as $x$ tends to $a$

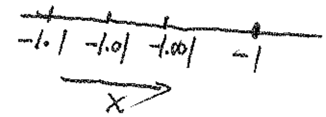
One-sided limit can be defined in the same way.

Remark1: "Infinite limit" is simply a notation. In this case, we still say "the limit does not exist".

Remark2: If one of the one-sided or two-sided limits at $x=a$ is $\pm\infty$, we say $x=a$ (Definition) is a $\boxed{\text{VERTICAL ASYMPTOTE}}$ of $y=f(x)$.

e.g.4. Compute the following limits and find the V.A. (vertical asymptote).

- $\lim\limits_{x\to 1^-} \frac{3}{x+1} = \frac{3}{1+1} = \frac{3}{2}$, (V.A.: $x=-1$)
- $\lim\limits_{x\to (-1)^-} \frac{3}{x+1} = -\infty$. Hint: $x\to(-1)^-$ indicates $x$ approaches $-1$ from the left. $x<-1$. $x+1$ small and negative.



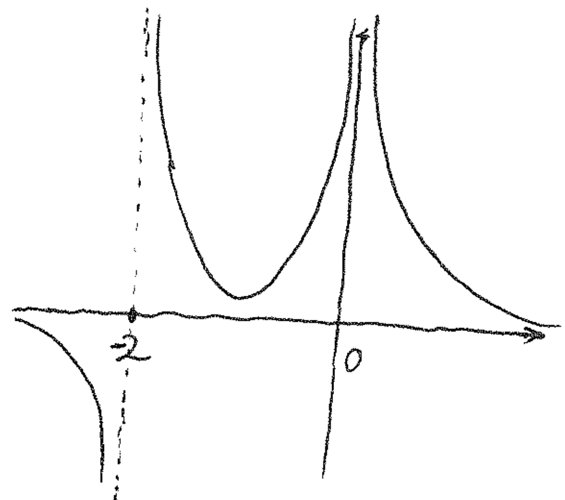

- $\lim\limits_{x\to 1^+} \frac{3}{(x-1)^2} = +\infty$. $x\to1^+ \Rightarrow x>1 \Rightarrow (x-1)^2$ small, positive.
- $\lim\limits_{x\to 1^-} \frac{-3}{(x-1)^2} = -\infty$. $x\to1^- \Rightarrow x<1 \Rightarrow x-1$ negative $\Rightarrow (x-1)^2$ positive $\Rightarrow \frac{-3}{(x-1)^2}$ negative.

e.g.5. Compute all the FOUR ONE-SIDED limits of $f(x)=\frac{1}{x^2(x+2)}$ at $x=0$, $x=-2$. (And sketch the graph of $f(x)$)

$\lim\limits_{x\to 0^-} \frac{1}{x^2(x+2)} = +\infty$, $\lim\limits_{x\to 0^+} \frac{1}{x^2(x+2)} = +\infty$

$\lim\limits_{x\to (-2)^-} \frac{1}{x^2(x+2)} = -\infty$. Hint: As $x\to-2^-$, $x^2$ positive
$x+2$ negative

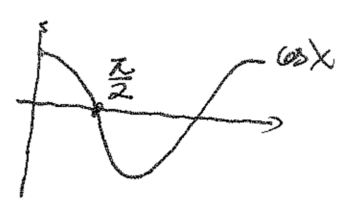
$\lim\limits_{x\to (-2)^+} \frac{1}{x^2(x+2)} = +\infty$ Hint: As $x\to(-2)^+$, $x^2$ positive
$x+2$ positive.




V.A.: $x=-2$ and $x=0$.

e.g.6. $\lim\limits_{x\to\frac{\pi}{2}} \sin x = 1$, $\lim\limits_{x\to\frac{\pi}{2}} \cos x = 0$.

$\lim\limits_{x\to(\frac{\pi}{2})^-} \frac{\sin x}{\cos x} = +\infty$. since $\cos x > 0$ for $x<\frac{\pi}{2}$.




$\lim\limits_{x\to(\frac{\pi}{2})^+} \frac{\sin x}{\cos x} = -\infty$ since $\cos x < 0$ for $x>\frac{\pi}{2}$.

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Hints for Webwork:

\#9: Use the fact $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\sec\theta = \frac{1}{\cos\theta}$

\#10: Complete the square for the denominator using $x^2+2ax+a^2=(x+a)^2$